

Universal scaling for the jamming transition

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The existence of universal scaling in the vicinity of the jamming transition of sheared granular materials is predicted by a phenomenology. The critical exponents are explicitly determined, which are independent of the spatial dimension. The validity of the theory is verified by the molecular dynamics simulation.

Jamming is an athermal phase transition between the solid-like jammed phase and the liquid-like unjammed phase of granular assemblies. Above the critical density, which is referred to the point J, the assemblies obtain the rigidity and the dynamic yield stress, while assemblies behave like dense liquids below the point J. Liu and Nagel¹⁾ indicated that the jamming transition is a key concept of glassy materials. Since then many aspects of similarities between the conventional glass transition and the jamming transition have been investigated.²⁾ Indeed, there are many examples where granular materials are used in order to investigate dynamical heterogeneity in glassy materials.^{3),4),5),6)} On the other hand, we still do not have an unified view in describing glassy materials because we cannot use the conventional theoretical tool for the glass transition such as the mode-coupling theory.⁷⁾

The jamming is a continuous transition that bulk and shear moduli become nonzero, and there are scaling laws in the vicinity of the point J similar to the cases of conventional critical phenomena.^{8),9)} Olsson and Teitel¹⁰⁾ and Hatano¹¹⁾ further demonstrated the existence of beautiful scalings near the point J. Therefore we can expect the existence of a simple theory in describing the jamming transition. However, we still do not have any theory to determine the critical exponents of the jamming transition.

In this letter, we predict the critical exponents for jamming transition of sheared granular materials based on a phenomenology. First, we introduce the system we consider and the scaling laws, some of which were introduced in Ref. 11). Second, based on a phenomenological theory, we decide the critical exponents for the jamming transition. Finally, we verify the theoretical prediction from our simulation. The most surprising in our finding is that the critical exponents are independent of the spatial dimension D , but depend on the type of particle interaction.

Let us consider a dense sheared and frictionless granular system in which uniform shear flow is stable. The system consists of N spherical grains in D dimensions. An

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important parameter to characterize the system is the volume fraction ϕ . In contrast to granular gases, the contact force plays crucially important roles in jamming transition, where the normal contact force is repulsive one characterized by $k\delta_{12}^\Delta$. Here, k is the stiffness constant, and $\delta_{12} = r - (\sigma_1 + \sigma_2)/2$ with the relative distance r between the contacting particles of the diameters σ_1 and σ_2 . We believe that Hertzian contact law $\Delta = 3/2$ is appropriate for three dimensional grains, but we often use a simpler linear spring model with $\Delta = 1$. In this letter, we omit any tangential contact force between grains. Thus, granular particles are frictionless, which can simplify the argument.

As introduced in Ref. 11), this system is expected to exhibit the scalings for the granular temperature T and the shear stress S in the sheared plane near the jamming transition as

$$T = A_{T,D} |\Phi|^{x_\Phi} \mathcal{T}_\pm \left(t_D \frac{\dot{\gamma}}{|\Phi|^{x_\Phi/x_\gamma}} \right), \quad (1)$$

$$S = A_{S,D} |\Phi|^{y_\Phi} \mathcal{S}_\pm \left(s_D \frac{\dot{\gamma}}{|\Phi|^{y_\Phi/y_\gamma}} \right), \quad (2)$$

where $\dot{\gamma}$ is the shear rate, $\Phi \equiv \phi - \phi_J$ is the excess volume fraction from the critical fraction ϕ_J at the point J, $A_{T,D}$ and $A_{S,D}$ are respectively amplitudes of the granular temperature and the shear stress. The scaling functions $\mathcal{T}_+(x)$ and $\mathcal{S}_+(x)$ above the point J respectively differ from $\mathcal{T}_-(x)$ and $\mathcal{S}_-(x)$ below the point J. It should be noted that $A_{T,D}$, $A_{S,D}$, t_D and s_D do not depend on $\dot{\gamma}$ and Φ , but depend only on D .

We also introduce a characteristic time scale of sheared granular assemblies as

$$\omega \equiv \frac{\dot{\gamma} S}{nT}, \quad (3)$$

where n is the number density of grains. This ω is reduced to the collision frequency in the unjammed phase in a steady state achieved by the balance between the viscous heating and the collisional energy loss. In contrast to the assumption by Hatano *et al.*,¹²⁾ ω also satisfies the scaling form

$$\omega = A_{w,D} |\Phi|^{z_\Phi} \mathcal{W}_\pm \left(w_D \frac{\dot{\gamma}}{|\Phi|^{z_\Phi/z_\gamma}} \right). \quad (4)$$

Thus, there are six critical exponents x_Φ , x_γ , y_Φ , y_γ , z_Φ , and z_γ in eqs. (1), (2) and (4). We note that the normal stress P also satisfies a similar scaling relation $P = A_{p,D} |\Phi|^{y'_\Phi} \mathcal{P}_\pm (p_D \dot{\gamma} / |\Phi|^{y'_\Phi/y'_\gamma})$,¹¹⁾ but we omit the details of the arguments on y'_Φ and y'_γ in this letter. We will discuss them elsewhere.

Below the point J, Bagnold's scaling should be held.^{13),14),15)} Thus, the scaling functions in the unjammed branch satisfy

$$\lim_{x \rightarrow 0} \mathcal{T}_-(x) = \lim_{x \rightarrow 0} \mathcal{S}_-(x) = x^2, \quad \lim_{x \rightarrow 0} \mathcal{W}_-(x) = x. \quad (5)$$

On the other hand, the jammed branch is characterized by the dynamic yield stress and the freezing of motion. Then, the scaling functions in the jammed branch satisfy

$$\lim_{x \rightarrow 0} \mathcal{S}_+(x) = \lim_{x \rightarrow 0} \mathcal{W}_+(x) = 1, \quad \lim_{x \rightarrow 0} \mathcal{T}_+(x) = x. \quad (6)$$

To obtain the last equation in eq. (6), we have used eq. (3) and the other two equations in eq. (6). Since the scaling functions are independent of Φ at the point J, we obtain

$$\lim_{x \rightarrow \infty} \mathcal{T}_{\pm}(x) \propto x^{x_{\gamma}}, \quad \lim_{x \rightarrow \infty} \mathcal{S}_{\pm}(x) \propto x^{y_{\gamma}}, \quad \lim_{x \rightarrow \infty} \mathcal{W}_{\pm}(x) \propto x^{z_{\gamma}}. \quad (7)$$

Now, let us determine the six critical exponents. First, we note that there are three trivial relations among the exponents. From eqs. (1)-(4) and (6) we obtain

$$z_{\Phi} = y_{\Phi} - x_{\Phi}(1 - x_{\gamma}^{-1}). \quad (8)$$

Similarly, from eqs.(1)-(4) with eq. (5) or (7) we respectively obtain

$$z_{\Phi}(1 - z_{\gamma}^{-1}) = y_{\Phi}(1 - 2y_{\gamma}^{-1}) - x_{\Phi}(1 - 2x_{\gamma}^{-1}), \quad (9)$$

$$z_{\gamma} = y_{\gamma} - x_{\gamma} + 1. \quad (10)$$

Thus, we further need three relations to determine the exponents.

In order to introduce the other relations, we consider the pressure P in the limit $\dot{\gamma} \rightarrow 0$. Let us consider Cauchy's stress in the jammed phase in which the pressure P is given by $P = \sum_{i>j} \langle F_{ij} r_{ij} \rangle / V$, where V is the volume of the system, r_{ij} and F_{ij} are respectively the distance and the force between i and j particles. This expression may be approximated by $P \simeq Z(\Phi) r_c(\Phi) F_c(\Phi)$ in the zero shear limit, where $Z(\Phi)$ is the average coordination number, $r_c(\Phi)$ and $F_c(\Phi)$ are respectively the average distance between contacting grains and the average force acting on the contact point. It is obvious that $Z(\Phi)$ and $r_c(\Phi)$ can be replaced by $Z(0)$ and $r_c(0) = \sigma$ in the vicinity of the jamming point, where σ is the average diameter of the particles. Indeed, O'Hern et al.⁹⁾ verified $Z(\Phi) - Z(0) \propto \Phi^{1/2}$ for three dimensional cases. Thus, the most important term is the mean contact force $F_c(\Phi) \propto \delta(\Phi)^{\Delta}$, where $\delta(\Phi)$ is the average length of compression. Now, let us compress the system at the critical point ϕ_J into $\Phi = \phi - \phi_J > 0$ by an affine transformation. Since all the characteristic lengths are scaled by the system size, we may assume the approximate relation $r_c(\Phi) = (\phi_J/\phi)^{1/D} \sigma$. From the relation $\delta(\Phi) = r_c(0) - r_c(\Phi)$, $\delta(\Phi)$ approximately satisfies $\delta(\Phi) \simeq (\sigma/D\phi_J)\Phi \sim \Phi$ in the vicinity of $\Phi = 0$. Thus, we conclude $P \sim \Phi^{\Delta}$. This relation has also been verified in Ref. 9).

On the other hand, it is well-known that there is Coulomb's frictional law in granular systems in which S/P is a constant. Indeed, Hatano¹⁶⁾ simulated the sheared granular system under a constant pressure P and demonstrated that the ratio satisfies $\lim_{\dot{\gamma} \rightarrow 0} S(\dot{\gamma}, P)/P = S_Y(P)/P = M_0$, where $S_Y(P) \equiv \lim_{\dot{\gamma} \rightarrow 0} S(\dot{\gamma}, P)$ and the constant M_0 is independent of the pressure P . The excess volume fraction $\Phi(\dot{\gamma}, P)$ in this system is a function of $\dot{\gamma}$ and P , but we also can express the pressure as $P(\dot{\gamma}, \Phi)$. Since $M_0 = \lim_{\dot{\gamma} \rightarrow 0} S(\dot{\gamma}, P(\dot{\gamma}, \Phi))/P(\dot{\gamma}, \Phi) = S_Y(P(0, \Phi))/P(0, \Phi)$ is independent of $P(0, \Phi)$, M_0 should be independent of Φ . Thus, we can conclude that Φ dependence of S is the same as that of P in the limit $\dot{\gamma} \rightarrow 0$. From this result, we obtain $y_{\Phi} = \Delta$ and

$$y_{\Phi} = 1 \quad \text{for linear spring model.} \quad (11)$$

This result also implies $y_{\Phi} = y'_{\Phi}$ which is consistent with the numerical observation.¹¹⁾

The next relation is related to the density of state. Wyart et al.¹⁷⁾ demonstrated the followings for unsheared assemblies of elastic soft spheres. (i) The jamming is related to the appearance of the soft modes in the density of state. (ii) There is a plateau in the density of state in the vicinity of jamming transition. (iii) The cutoff frequency ω^* of the plateau is proportional to \sqrt{P} . From the argument in the previous paragraph, the applied pressure satisfies the relation $P \propto \Phi^\Delta$. When we assume that the characteristic frequency ω in the limit $\dot{\gamma} \rightarrow 0$ can be scaled by the cutoff frequency ω^* , we may conclude $\omega \sim |\Phi|^{1/2}$ for the linear spring model. Thus, we obtain $z_\Phi = \Delta/2$ or

$$z_\Phi = 1/2 \quad \text{for linear spring model.} \quad (12)$$

Finally, we consider the characteristic frequency ω in the unjammed phase ($\Phi < 0$). In this phase, the characteristic frequency ω is estimated as $\omega \sim \sqrt{T/m}/l(\Phi)$, where $l(\Phi)$ is the mean free path. Note that $l(\Phi)$ may be evaluated as $(\sigma/D\phi_J)|\Phi|$ in the vicinity of the point J, using the parallel argument to $\delta(\Phi)$ for $\Phi > 0$. From the scalings in Bagnold's regime (5), we obtain $\omega \sim |\Phi|^{z_\Phi(1-z_\gamma^{-1})}\dot{\gamma}$ and $T \sim |\Phi|^{x_\Phi(1-2x_\gamma^{-1})}\dot{\gamma}^2$. Substituting these relations to $\omega \sim \sqrt{T/m}/l(\Phi)$, we obtain

$$z_\Phi(1 - z_\gamma^{-1}) - \frac{1}{2}x_\Phi(1 - 2x_\gamma^{-1}) = -1. \quad (13)$$

From the above six relations (8)-(13) we finally determine the six critical exponents

$$\begin{aligned} x_\Phi &= 3, & x_\gamma &= \frac{6}{5}, & y_\Phi &= 1, \\ y_\gamma &= \frac{2}{5}, & z_\Phi &= \frac{1}{2}, & z_\gamma &= \frac{1}{5} \end{aligned} \quad (14)$$

for the linear spring model. The exponents in Hertzian model are, of course, different. In general situation for Δ , eqs. (14) are replaced by

$$\begin{aligned} x_\Phi &= 2 + \Delta, & x_\gamma &= \frac{2\Delta + 4}{\Delta + 4}, & y_\Phi &= \Delta, \\ y_\gamma &= \frac{2\Delta}{\Delta + 4}, & z_\Phi &= \frac{\Delta}{2}, & z_\gamma &= \frac{\Delta}{\Delta + 4}. \end{aligned} \quad (15)$$

We should note that the exponents are independent of the spatial dimension. This is not surprising because our phenomenology to derive eqs. (11) and (12) is independent of the spatial dimension.^{9),17)} We should stress an interesting feature of jamming transition that the exponents strongly depend on the interaction model among particles. This property is contrast to that in the conventional critical phenomena. Thus, we should be careful to use the idea of the universality in describing the jamming transition.

From now on, let us verify our theoretical results based on the molecular dynamics simulation. In our simulation, the system consists of N spherical grains in 2, 3, 4 dimensions. We adopt the linear spring model ($\Delta = 1$) for simplicity. We also

introduce dissipative force $-\eta\delta v$, where δv represents the relative velocity between the contacting particles. Each grain has an identical mass m . In order to realize an uniform velocity gradient $\dot{\gamma}$ in y direction and macroscopic velocity only in the x direction, we adopt the Lees-Edwards boundary conditions. The particle diameters are $0.7\sigma_0$, $0.8\sigma_0$, $0.9\sigma_0$ and σ_0 each of which is assigned to $N/4$ particles.

In our simulation m , σ_0 and η are set to be unity, and all quantities are converted to dimensionless forms, where the unit of time scale is m/η . We use the spring constant $k = 1.0$. For the system near the critical density, such as $\phi = 0.8428$ for $D = 2$, $\phi = 0.643$ and 0.6443 for $D = 3$, we use $N = 4000$ in order to remove finite size effects. For other systems, we use $N = 2000$.

The scaling plots of our simulation based on the exponents (14) are shown in Fig. 1. We should stress that these scaling plots contain the data in $D = 2, 3$ and 4. The volume fraction at the point J is estimated as $\phi_J = 0.84285$ for $D = 2$, $\phi_J = 0.64455$ for $D = 3$ or $\phi_J = 0.4615$ for $D = 4$. We examine the shear rate $\dot{\gamma}$ is in the range between 5×10^{-7} and 5×10^{-5} for $D = 2, 3$ and between 5×10^{-6} and 5×10^{-4} for $D = 4$. The amplitudes and the adjustable parameters are obtained as $(t_D, A_{t,D}, s_D, A_{s,D}, w_D, A_{w,D}) = (0.0125, 7.17, 0.025, 0.035, 0.05, 0.3)$ for $D = 2$, $(0.01385, 2.527, 0.03, 0.04, 0.065, 0.65)$ for $D = 3$, $(0.015, 1.6275, 0.03, 0.06, 0.06, 1)$ for $D = 4$. Since Fig.1 exhibits beautiful scaling laws, our phenomenology seems to be right.

Figure 2(a) shows Φ dependence of the shear viscosity $\mu \equiv S/\dot{\gamma}$ in Bagnold's regime. We should note that there is no consensus in previous studies on the shear viscosity. For example, Garcia-Rojo et al.¹⁸⁾ reported $\mu \sim 1/(\phi_c - \phi)$, where ϕ_c is lower than ϕ_J , while Losert et al.¹⁹⁾ observed the exponent larger than 1 from their experiment, and the exponent of divergence in Ref. 10) is also between 1 and 2. We also note that the viscosity is believed to diverge as $|\Phi|^{-2}$ for colloidal suspensions.²⁰⁾ However, our scaling theory predicts $\mu \propto |\Phi|^{y_\phi(1-2/y_\gamma)} \propto |\Phi|^{-4}$, and the viscosity diverges at the point J. Here, the scaling exponents for μ is independent of Δ because y_ϕ and y_γ for an arbitrary Δ are determined from our theory as in eq. (15). As we can see in Fig. 2(a), the theoretical prediction is consistent with our numerical result. We also examine the possibility that the viscosity diverges at $\phi_c < \phi_J$ with $\mu \sim (\phi_c - \phi)^{-1}$ as in the case of Garcia-Rojo et al.¹⁸⁾ Actually we can fit the data of our two-dimensional simulation by $\mu \sim (\phi_c - \phi)^{-1}$ with $\phi_c = 0.835$ which is less than $\phi_J = 0.8428$ for $\phi < \phi_c$, but the viscosity is still finite even for $\phi > \phi_c$ (see Fig. 2(b)). Thus, we can conclude that (i) the viscosity does not satisfy $(\phi_c - \phi)^{-1}$ but exhibits a consistent behavior with $(\phi_J - \phi)^{-4}$ predicted by our phenomenology, and (ii) the critical behaviors are only characterized by the point J.

We also verify the validity of $\omega \sim |\Phi|^{1/2}$ in the jammed phase from our simulation in Fig. 3. The envelope line of our result seems to be consistent with the theoretical prediction.

It should be noted that the existence of the plateau for $|\Phi| \rightarrow 0$ in Fig. 2(a) can be understood from the scaling relation (2). Indeed, Φ dependence of the shear stress disappears in the limit of large $x \equiv \dot{\gamma}/|\Phi|^{y_\phi/y_\gamma}$ with $\mathcal{S}_-(x) \rightarrow x^{y_\gamma}$ as in eq.(7). Thus, we obtain $\mu/\dot{\gamma} = S/\dot{\gamma}^2 \sim \dot{\gamma}^{y_\gamma-2} \sim \dot{\gamma}^{-8/5}$. This estimation might be consistent with the simulation in Fig. 2(a) in which the value of the plateau increases as the

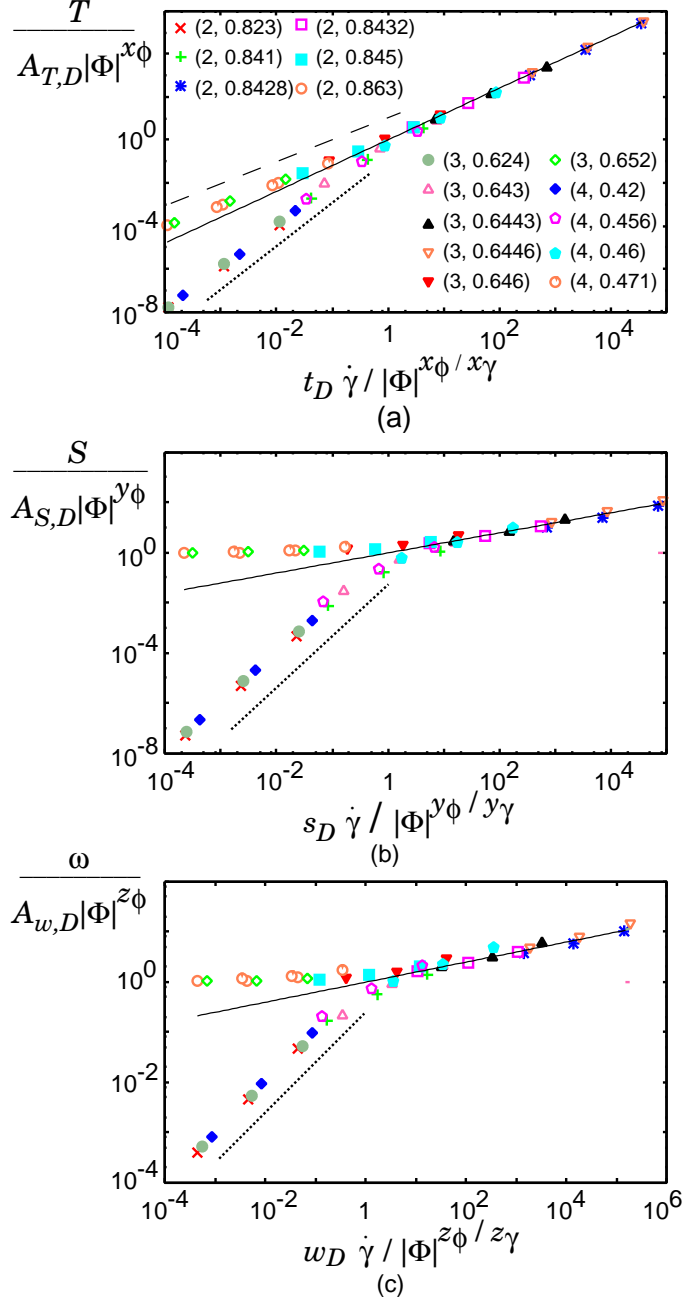


Fig. 1. (Color online) (a): Collapsed data of the shear rate dependence of the granular temperature T using the scaling law for $D = 2, 3$ and 4 . The dashed line, the dotted line and the solid line are proportional to $\dot{\gamma}$, $\dot{\gamma}^2$ and $\dot{\gamma}^{x_\gamma}$. The legends show the dimension D and the volume fraction ϕ as (D, ϕ) . (b): Collapsed data of the shear rate dependence of the shear stress S using the scaling law for $D = 2, 3$ and 4 . The dotted line and the solid line are proportional to $\dot{\gamma}^2$ and $\dot{\gamma}^{y_\gamma}$. (c): Collapsed data of the shear rate dependence of the cooling rate ω using the scaling law for $D = 2, 3$ and 4 . The dotted line and the solid line are proportional to $\dot{\gamma}^2$ and $\dot{\gamma}^{z_\gamma}$.

shear rate decreases. Similarly, we can expect the value of plateau of ω as $\dot{\gamma}^{z_\gamma}$, while this saturation cannot be verified from the simulation.

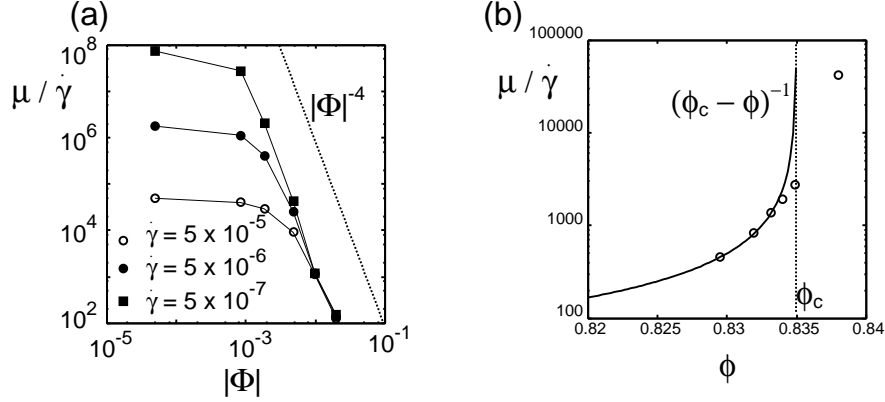


Fig. 2. (a) : $\mu/\dot{\gamma}$ as a function of Φ for $D = 2$ with $\dot{\gamma} = 5 \times 10^{-5}, 5 \times 10^{-6}, 5 \times 10^{-7}$ in the unjammed phase. (b) : $\mu/\dot{\gamma}$ as a function of ϕ for $\dot{\gamma} = 5 \times 10^{-7}$ in the unjammed phase, where the solid line is proportional to $(\phi_c - \phi)^{-1}$ with $\phi_c = 0.835$.

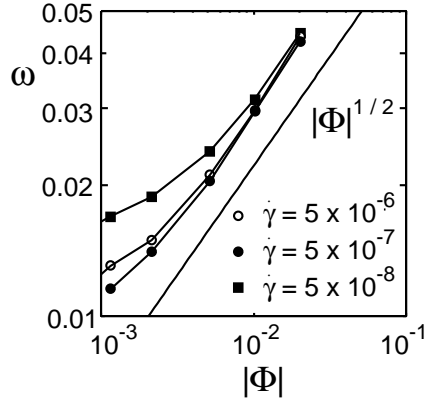


Fig. 3. ω as a function of Φ for $D = 2$ with $\dot{\gamma} = 5 \times 10^{-6}, 5 \times 10^{-7}, 5 \times 10^{-8}$ in the jammed phase.

Now, let us discuss our results. First of all, the ratios between the exponents x_ϕ/x_γ , y_ϕ/y_γ , and z_ϕ/z_γ obtained in eq. (14) or (15) satisfy

$$\alpha \equiv \frac{x_\phi}{x_\gamma} = \frac{y_\phi}{y_\gamma} = \frac{z_\phi}{z_\gamma} = \frac{\Delta + 4}{2}. \quad (16)$$

This is not surprising because the time scale is expected to be scaled by the shear rate. Thus, the ratio α in eqs. (1), (2), and (4) should be common. In other words, the characteristic time scale τ exhibits the critical slowing down as $\tau \sim |\Phi|^{-\alpha}$. This property has already been indicated by Hatano.¹¹⁾ Once we accept the ansatz (16), eqs. (8), (9), and (10) are degenerate, and reduce to

$$x_\phi - y_\phi + z_\phi = \alpha. \quad (17)$$

Equation (13) is also reduced to the simplified form

$$z_\phi = \frac{x_\phi}{2} - 1. \quad (18)$$

From these equations and eqs. (11) and (12) with (16), we obtain eq. (14) or eq. (15).

Second, Hatano estimated the exponents $x_\phi = 2.5$, $x_\gamma = 1.3$, $y_\phi = 1.2$ and $y_\gamma = 0.57$ from his three-dimensional simulation for the linear spring model,¹¹⁾ which differ from our prediction (14). In particular, if we use these values with eq. (18), z_ϕ is estimated as $z_\phi = 0.25$, which is one half of our prediction $z_\phi = 1/2$. However, the estimation of the scaling exponents strongly depend on the choice of ϕ_J and the range of the shear rate $\dot{\gamma}$. The value of ϕ_J and the range of $\dot{\gamma}$ in Ref. 11) are larger than ours. If we adopt Hatano's ϕ_J and the range of $\dot{\gamma}$, our numerical data can be scaled by Hatano's scaling. Although his scaling can be used in the wide range of $\dot{\gamma}$, the deviation from his scaling can be detected in the small $\dot{\gamma}$ region ($\dot{\gamma} < 10^{-4}$). It is obvious that we should use smaller $\dot{\gamma}$ as possible as we can to extract the critical properties. This suggests that our exponents are more appropriate than Hatano's exponents in characterizing the jamming transition. The difficulty in determination of the exponents from the simulation also supports the significance of our theory to determine the scaling laws.

Third, the exponents obviously depend on the model of interaction between particles as predicted in eq. (15). Our preliminary simulation suggests that the numerical exponents for Hertzian contact model is consistent with the prediction of (15). The numerical results on Δ dependence of the exponents will be reported elsewhere.

Fourth, our results should be modified when we analyze the model in the zero temperature limit of Langevin thermostat. This situation corresponds to that in Ref. 10). In this case, we should replace Bagnold's law in unjammed phase by Newtonian law $S \propto \dot{\gamma}$. As a result, all the scaling exponents have different values. We will discuss the results of this situation elsewhere.

Finally, we comment on the relation between our results and the previous studies on dynamical heterogeneity in glassy materials. The dynamical heterogeneity in glassy materials is characterized by the large fluctuations of four point correlation function, in which the result strongly depends on the spatial dimension. On the other hand, our theory and numerical simulation suggest that the critical fluctuation is not important and our phenomenology works well. Since the quantities we analyzed in this letter are not directly related to the four-point correlation functions, there is no distinct contradiction between them. To study the roles of critical fluctuations and dynamical heterogeneity we may need a more sophisticated theory. This will be our future task.

In conclusion, we develop the phenomenological theory in describing the jamming transition. We determine the critical exponents which are independent of the spatial dimension. The validity of our theory has been verified by the molecular dynamics simulation.

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